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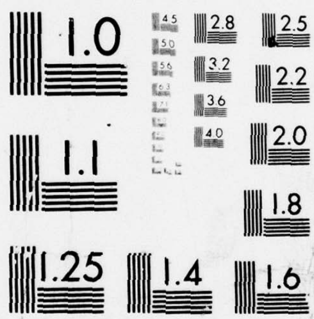

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $T_1,\ldots,T_n$ be positive random variables with finite means. Further let I be the collection of all subsets of $\{1,\ldots,n\}$, and let ξ be a function from the nth Euclidian space to I, that equals to J, $(J \in I)$ at $(a_1,\ldots,a_n)$ iff $\min_{i\in J} a_iT_i < \min_{i\notin J} a_iT_i$. We prove that $\min_{1\le i\le n} a_iT_i$ and $\xi(a_1,\ldots,a_n)$ are independent random variables for every n real numbers $a_1,\ldots,a_n$ iff for every n positive

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the minimum of (a sub i)(T sub i) over i from 1 to n

20. Abstract continued.

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real numbers $b_1,\ldots,b_n$ and $r = 1,\ldots,n$ the random variables $\min\limits_{1\le i\le n} a_iT_i/E(\min\limits_{1\le i\le n} a_iT_i)$ and T_r/ET_r are identically distributed. Further we

sub r sub 1 sub n

provide an explicit formula for the distribution of $\xi(a_1,\ldots,a_n)$. Multivariate distributions that possess the independence property are presented. Their use in Reliability growth or decay models as well as in Mathematical Epidemiology are discussed.

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ON A CHARACTERIZATION OF MULTIVARIATE DISTRIBUTIONS WITH APPLICATIONS IN RELIABILITY AND EPIDEMIOLOGY

by

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On a Characterization of Multivariate Distributions with Applications in Reliability and Epidemiology

by

Naftali A. Langberg

ABSTRACT

Let $T_1, \ldots, T_n$ be positive random variables with finite means. Further let I be the collection of all subsets of $\{1, \ldots, n\}$, and let ξ be a function from the nth Euclidian space to I, that equals to J, ($J \in I$) at $(a_1, \ldots, a_n)$ iff $\min_{i \in J} a_i T_i < \min_{i \notin J} a_i T_i$. We prove that $\min_{1 \le i \le n} a_i T_i$ and $\xi(a_1, \ldots, a_n)$ are independent random variables for every n real numbers $a_1, \ldots, a_n$ iff for every n positive real numbers $b_1, \ldots, b_n$ and $r = 1, \ldots, n$ the random variables $\min_{1 \le i \le n} a_i T_i / E(\min_{1 \le i \le n} a_i T_i)$ and T_r/ET_r are identically distributed. Further we provide an explicit formula for the distribution of $\xi(a_1, \ldots, a_n)$. Multivariate distributions that possess the independence property are presented. Their use in Reliability growth or decay models as well as in Mathematical Epidemiology are discussed.

Key Words: Multivariate distribution, minima, Weibull, distribution, independence

1. Introduction and Summary.

Introduction. Let $T_1, \ldots, T_n$ be positive random variables, and let $a_1, \ldots, a_n$ be positive real numbers. If $T_1, \ldots, T_n$ are the initial life lengths of n components in a series system, then $a_1T_1, \ldots, a_nT_n$ can be regarded as the life lengths of those components at some phase of a reliability growth or decay process. Let an individual be exposed to n contagious diseases in an environment consisting of infectives and susceptibles. Then $a_1T_1, \ldots, a_nT_n$ may describe the times until that individual becomes an infective from disease 1 through n respectively. If we observe a series system, or an individual who is exposed to n diseases, only two quantities are identifiable: (i) Time until occurence (failure, or infection) (ii) Cause of occurence (failure due to some components, infection by some of the diseases). The stochastic representation and analysis of the described models simplifies if for every n positive real numbers the following two properties hold.

Time to occurence and cause of occurence are independent random variables. (1.1)

Time to occurence and cause of occurence have "identifiable" distributions. (1.2)

Recently Langberg, Lanzdorf and Proschan (1978) used multivariate distributions that satisfy (1.1) and (1.2) to describe and analyze a variety of reliability growth and decay models. In the cited reference the authors considered multivariate distributions with independent and dependent components, as well as distributions that may or may not be absolute continuous. A well known family of n-dimensional random vectors with independent components that satisfy (1.1) and (1.2) is the exponential one. More specifically let $T_1, \ldots, T_n$ be independent exponential random variables with means $\mu_1, \ldots, \mu_n$ respectively, and let $a_1, \ldots, a_n$ be positive real numbers. Then the following three statements hold.

Time to occurance is exponential with mean equal to $(\sum_{j=1}^{n} a_j^{-1}\mu_j^{-1})^{-1}$. (1.3)

The probability of occurence due to cause i equals to $(a_i^{-1}\mu_i^{-1})(\sum_{j=1}^{n} a_j^{-1}\mu_j^{-1})^{-1}$ for $i = 1, \ldots, n$. (1.4)

Time to occurence and cause of occurence are independent random variables. (1.5)

Billard, Lacayo and Langberg (1978), Lacayo, Langberg (1978) and Langberg (1978), utilized those properties of independent exponential random variables to describe and analyze n-dimensional simple epidemics.

Summary: Lemmas instrumental to the proofs of our main results are presented in Section 2. In Section 3 we show that condition (1.1) is equivalent to the following statement.

For every n positive real numbers $a_1, \ldots, a_n$, $(E \min_{1\le i\le n} a_i T_i)^{-1} \min_{1\le i\le n} a_i T_i$ and $(ET_j)^{-1} T_j$ are identically distributed random variables for $j = 1, \ldots, n$. (1.6)

Clearly (1.6) provides an explicit formula for the distribution of the random time until occurence. In addition we derive in Section 3 from (1.6) an explicit form for the distribution of the cause to occurence. Some multivariate distributions which satisfy (1.6) are presented in the last Section.

2. Preliminaries.

Let denote by (T_1, T_2) a positive random vector with means equal to μ_1, μ_2 respectively, by $T(x)$ the minimum of $x\,T_1$, and T_2, and by $F(x)$ the expected value of $T(x)$. Further let $F'(x+)$ and $F'(x-)$ be the right and left side derivatives of F at the point x. Finally let $G(x)$ be equal to $\mu_2^{-1} I(x \geq 0)F(x)$, $G_n(x)$ be the convolution of G and a uniform distribution on $[-\frac{1}{n}, 0]$, and let $g_n(t)$ be the density function of $G_n(t)$, $n = 1, 2, \ldots$. For reference purposes we summarize in Lemma 2.1 without proofs some straight forward results.

<u>Lemma 2.1</u>. The following six statements hold.

F is a concave nondecreasing function. (2.1)

$$\operatorname*{Lim}_{x \to 0^+} F(x) = 0, \text{ and } \operatorname*{Lim}_{x \to \infty} F(x) = \mu_2. \tag{2.2}$$

$$F'(x+) = ET_1 I(T_2 > xT_1) \text{ and } F'(x-) = ET_2 I(T_2 \geq xT_1). \tag{2.3}$$

$$\operatorname*{Lim}_{n \to \infty} \operatorname*{Sup}_{x} |G_n(x) - G(x)| = 0. \tag{2.4}$$

g_n is nondecreasing in n, and $\operatorname*{Lim}_{n \to \infty} g_n(x) = G'(x+)$, for x in $(0, \infty)$. (2.5)

$G_n(x) \geq G(x)$, for $n = 1, 2, \ldots$, and every real number x. (2.6)

<u>Lemma 2.2</u>. Let x and y be real positive numbers, then

$$\int_x^y \frac{F'(u+)}{F(u)}\, du = \ell nF(y) - \ell nF(x). \tag{2.7}$$

<u>Proof</u>. Since by (2.6) $G_n(x)$ and $G_n(y)$ are positive real numbers, we obtain that $\int_x^y \frac{g_n(u)}{G_n(u)} du = \ell nG_n(y) - \ell_n G_n(x)$. Statement (2.7) follows from (2.4), (2.5) and the monotone convergence theorem.

Let $\xi(x)$ be the cause of occurence function given by

$$\{1\}I(T_2 > xT_1) + \{2\}I(T_2 < xT_1) + \{1,\ 2\}I(T_2 = xT_1). \tag{2.8}$$

Lemma 2.3. If $\xi(x)$ and $T(x)$ are independent random variables for every real number x, then for z in $(0, \infty)$

$$F(z) = \mu_2 \exp[-\int_z^{\infty} \frac{P(T_2 > uT_1)}{u} du]. \tag{2.9}$$

Proof. By (2.3) and the independence assumption $F'(z+)$ equals to $z^{-1}F(z)P(T_2 > zT_1)$. Equation (2.9) is obtained from (2.2) and (2.7).

Theorem 2.4. Let $\xi(x)$ and $T(x)$ be independent random variables for every real number x. Further let's assume that $\overline{\lim_{k\to\infty}}\, k^{-1}(ET_2^k)^{1/k}$ is finite. Then

$$T(x) \text{ and } \mu_2^{-1}F(x)T_2 \text{ are identically distributed for } x \text{ in } (0, \infty). \tag{2.10}$$

Proof. From (2.9) we obtain that $E\,T^k(x) = E[\mu_2^{-1}F(x)T_2]^k$ for $k = 1, 2, \ldots$. Consequently (2.10) follows from the moments property of T_2. [Breiman (1968), pp. 182, proposition 8.49].

Remark 2.5. Since $x \min(x^{-1}T_1, T_2) = \min(T_1, xT_2)$, it follows that if the conditions of Theorem (2.4) are satisfied, then

$$\mu_1^{-1}T_1 \text{ and } \mu_2^{-1}T_2 \text{ are identically distributed.} \tag{2.11}$$

Lemma 2.6. Let T_1 and T_2 have positive continuous densities f_1 and f_2 respectively. Further let x and y be positive real numbers, and $a = yx^{-1}$. If (2.10) is satisfied, then the following two equations hold.

$$P\{T_2 > y | T_1 = x\} = f_1^{-1}(x) f_2(y\, F^{-1}(a)) a^2\, F^{-2}(a) F'(a+). \tag{2.12}$$

$$P\{T_2 \geq y | T_1 = x\} = f_1^{-1}(x) f_2(y\, F^{-1}(a)) a^2\, F^{-2}(a) F'(a-). \tag{2.13}$$

Proof. Equation (2.13) follows clearly from (2.12). To prove (2.12) it suffices to consider the case $F'(a+) = F'(a-)$. A proof for this case involves elementary calculations and is omitted.

Remark 2.7. Let Z be a positive random variable independent of (T_1, T_2). If the vector (T_1, T_2) satisfies (2.10), then so does $(T_1 Z, T_2 Z)$.

Theorem 2.8. If (2.10) holds, then for every real number x the random variables $\xi(x)$ and $T(x)$ are independent.

Proof. Firstly let's assume that T_1 and T_2 have positive and continuous densities f_1 and f_2 respectively. Since $P[T(x) > t, \xi(x) = \{1\}] = \int_{t/x}^{\infty} P[T_2 > xu | T_1 = u] f_1(u) du$, and $P[T(x) > t, \xi(x) = \{2\}] = \int_{t/x}^{\infty} P[xu > T_2 > t | T_1 = u] f_1(u) du$, the result for this particular case follows from (2.12) and (2.13). To complete the proof, let $Z_1, Z_2, \ldots,$ be a sequence of random variables independent of (T_1, T_2), given by $P[Z_n \leq t] = [1 - e^{-n(t-1)}] I(t \geq 1)$, $n = 1, 2, \ldots$. $T_1 Z_n$, $T_2 Z_n$ have continuous positive densities and by Remark (2.7) satisfy condition (2.10) for $n = 1, 2, \ldots$. Consequently $Z_n T(x)$ and $\xi(x)$ are independent for every real number x, and positive integer n. Since Z_n converges in probability to 1 as $n \to \infty$ the desired result follows.

Finally we note that if $\xi(x)$ and $T(x)$ are independent for every real number x, then by (2.3) the following three equations hold for z in $(0, \infty)$.

$$P[T_2 > z\ T_1] = z\ F'(z+)/F(z). \tag{2.14}$$

$$P[T_2 < z\ T_1] = 1 - z\ F'(z-)/F(z). \tag{2.15}$$

$$P[T_2 = z\ T_1] = z[(F'(z-) - F^1(z+)]/F(z). \tag{2.16}$$

3. Main Results.

Let $(T_1, \ldots, T_n)$ be a positive random vector with means equal respectively to $\mu_1, \ldots, \mu_n$, and let I be the set of all nonempty subsets of $\{1, \ldots, n\}$. Further let $T(a_1, \ldots, a_n)$ be $\min_{1 \le i \le n} a_i T_i$, and let $F(a_1, \ldots, a_n)$ be the expected value of $T(a_1, \ldots, a_n)$. We define the cause of occurence function $\xi(a_1, \ldots, a_n)$ as

$$\sum_{J \varepsilon I} j I(\min_{i \varepsilon J} a_i T_i < \min_{i \not\varepsilon J} a_i T_i). \tag{3.1}$$

Finally let denote by statement (3.2) the following property of $(T_1, \ldots, T_n)$.

There exists an integer i_0 in $\{1, \ldots, n\}$, such that $T(a_1, \ldots, a_n)$ and $T_{i_0} \mu_{i_0}^{-1} F(a_1, \ldots, a_n)$ are identically distributed for every n positive real numbers $a_1, \ldots, a_n$. (3.2)

For reference purposes we note that

<u>Lemma 3.1</u>. If (3.2) is satisfied, then the following three statements hold.

$\mu_r^{-1} T_r$, $r = 1, \ldots, n$ are identically distributed random variables. (3.3)

For $J \varepsilon I$ and a_i, $i \varepsilon J$ positive real numbers, $\min_{i \varepsilon J} a_i T_i$ and $\mu_r^{-1}(E \min_{i \varepsilon J} a_i T_i)^{-1} T_r$ are identically distributed random variables for $r = 1, \ldots, n$. (3.4)

For $J \varepsilon I$ and $a_1, \ldots, a_n$ positive real numbers, $\min_{1 \le i \le n} a_i T_i$ and $(\min_{i \varepsilon J} a_i T_i) F(a_1, \ldots, a_n)(E \min_{i \varepsilon J} a_i T_i)^{-1}$ are identically distributed random variables. (3.5)

We are ready to extend Theorems 2.4 and 2.7.

<u>Theorem 3.2</u>. Let $\xi(a_1, \ldots, a_n)$ and $T(a_1, \ldots, a_n)$ be independent random variables for every n real numbers $a_1, \ldots, a_n$. Further let assume that $\overline{\lim}_{k \to \infty} k^{-1}(E\, T_r^k)^{1/k}$ is finite for some positive integer r in $\{1, \ldots, n\}$. Then (3.2) holds.

Proof. Let $\overline{\lim}_{k\to\infty} k^{-1}(E\, T_{i_0}^k)^{1/k}$ be finite, and let U_1 and U_2 be respectively equal to $\min_{i\neq i_0} a_i T_i$ and $a_{i_0} T_{i_0}$. Since (U_1, U_2) satisfies the conditions of Theorem (2.4), statement (3.2) follows.

Theorem 3.3. If (3.2) holds, then for every n real numbers $a_1, \ldots, a_n$, the random variables $\xi(a_1, \ldots, a_n)$ and $T(a_1, \ldots, a_n)$ are independent.

Proof. Let $J \varepsilon I$ and U_1, U_2 be equal respectively to $\min_{i\varepsilon J} a_i T_i$ and $\min_{i\notin J} a_i T_i$. Since by (3.5) condition (2.10) is satisfied, the result is obtained by Theorem 2.7.

Remark 3.4. Let J be in I, further let U_1, U_2 be respectively equal to $\min_{i\varepsilon J} a_i T_i$ and to $\min_{i\notin J} a_i T_i$. Finally let $H(x)$ be equal to $E \min(xU_1, U_2)$. If $(T_1, \ldots, T_n)$ satisfies (3.2), then by (2.3)

$$P\{\xi(a_1, \ldots, a_n) = J\} = H'(x+)\big|_{x=1} \cdot F(a_1, \ldots, a_n). \tag{3.6}$$

Remark 3.5. Let Z be a positive random variable, independent of $(T_1, \ldots, T_n)$. If $(T_1, \ldots, T_n)$ satisfies (3.2), then so does $(T_1 Z, \ldots, T_n Z)$.

A nonnegative random variable T has a Weibull distribution with parameters μ and α if for t in $(0, \infty)$

$$P\{T > t\} = \exp[-\mu t^{\alpha}], \ \mu, \alpha > 0. \tag{3.7}$$

Let α be a positive real number and g a positive real function defined on the nth Euclidian space. We say that a nonnegative random vector $(T_1, \ldots, T_n)$ has Weibull minima, if for every n positive real numbers $a_1, \ldots, a_n$ and t in $(0, \infty)$

$$P\{T(a_1, \ldots, a_n) > t\} = \exp[-g(a_1, \ldots, a_n) t^{\alpha}]. \tag{3.8}$$

Clearly every subset of n random variables with Weibull minima has Weibull minima. If T is a Weibull random variable with parameters μ and α then

$$E\,T^k = \mu^{-k/\alpha} \int_0^\infty e^{-z} z^{k/\alpha} dz. \tag{3.9}$$

Consequently if T is Weibull with $\alpha \geq 1$ then

$$\overline{\lim_{k\to\infty}}\, k^{-1} (E\,T^k)^{1/k} < \infty \tag{3.10}$$

and the following corollary holds:

Corollary 3.6. (i) If $(T_1, \ldots, T_n)$ has Weibull minima then $\xi(a_1, \ldots, a_n)$ and $T(a_1, \ldots, a_n)$ are independent random variables for every n positive real numbers $a_1, \ldots, a_n$. (ii) If at least one of the T_i's is Weibull with $\alpha \geq 1$, and $\xi(a_1, \ldots, a_n)$ is independent of $T(a_1, \ldots, a_n)$ for every n real numbers $a_1, \ldots, a_n$, then $(T_1, \ldots, T_n)$ has Weibull minima.

Proof. Follows from Theorems 3.2, 3.3, and from (3.10).

Essary and Marshall (1974) defined a class of positive n-dimensional random vectors that have the following property.

For $J \in I$ and a_i, $i \in J$ positive real numbers, $\min_{i\in J} a_i T_i$ has an exponential distribution. (3.11)

Corollary 3.6 provides in particular the following characterization of that class.

Corollary 3.7. Let $(T_1, \ldots, T_n)$ be a positive random vector with at least one exponential component. $(T_1, \ldots, T_n)$ satisfies (3.11) iff $\xi(a_1, \ldots, a_n)$ and $T(a_1, \ldots, a_n)$ are independent random variables for every n real numbers $a_1, \ldots, a_n$.

Let $R(t)$ be a positive, strictly increasing function, that is differentiable and converges to ∞ as $t \to \infty$. Further let h be a positive real function on the nth Euclidian space. We say that $(T_1, \ldots, T_n)$ has proportional minima, if for every n positive real numbers $a_1, \ldots, a_n$ and t in $(0, \infty)$

$$P\{T(a_1, \ldots, a_n) > t\} = \exp[-h(a_1, \ldots, a_n)R(t)]. \tag{3.12}$$

Finally we show that the only family of random vectors with proportional minima is the Weibull one.

Theorem 3.10. Let $(T_1, \ldots, T_n)$ be a positive random vector. $(T_1, \ldots, T_n)$ $(n \geq 2)$ has proportional minima iff it has Weibull minima.

Proof. Let $\theta(a)$ be $h(a, \ldots, a)$, then $R(ta)\theta(a) = R(t)\theta(1)$. Consequently $\frac{t}{R}\frac{dR}{dt} = \frac{a}{h}\frac{dh}{da}$, hence $R(t) = At^c$ for some positive real numbers A and c.

4. Examples.

Example 4.1. Let λ_J, $J \varepsilon I$ be nonnegative real numbers, that add up over the sets in I to a positive number. We define the multivariate distribution of $(T_1, \ldots, T_n)$ for $t_1, \ldots, t_n$ in $(0, \infty)$ by

$$P\{T_i > t_i,\ i = 1, \ldots, n\} = \exp[-\sum_{J\varepsilon I} \lambda_J \max_{i\varepsilon J} t_i^{\alpha}],\ \alpha > 0. \tag{4.1}$$

The distribution in (4.1) is an extension of the bivariate Marshall-Olkin (1967) exponential distribution. Clearly this family of random vectors has Weibull minima with $g(a_1, \ldots, a_n) = \sum_{J\varepsilon I} \lambda_J \max_{i\varepsilon J} a_i^{-\alpha}$. In particular we obtain for $\lambda_J = 0$, whenever J is not a singlton set independent Weibull random variables. If $\lambda_J = 0$ for $J \neq \{1, \ldots, n\}$ the multivariate distribution given by (4.1) reduces to correlated Weibull random variables.

Example 4.2. Let λ_1, λ_2 be nonnegative real numbers. Further let $\gamma_1, \gamma_2 \ldots$ be a sequence of positive numbers that add up to a finite real number. Finally let $b_1, b_2 \ldots$ be a sequence of positive real numbers. We define the bivariate distribution (T_1, T_2) for t_1, t_2 and α in $(0, \infty)$ by

$$P\{T_i > t_i,\ i = 1, 2\} = \exp[-\lambda_1 t_1^{\alpha} - \lambda_2 t_2^{\alpha} - \sum_{n=1}^{\infty} \lambda_n \max(t_1^{\alpha}, t_2^{\alpha} b_n^{-\alpha})]. \tag{4.2}$$

This distribution has Weibull minima with $g(a_1, a_n)$ equal to $\lambda_1 a_1^{-\alpha} + \lambda_2 a_2^{-\alpha} + \sum_{n=1}^{\infty} \lambda_n (a_1^{-\alpha} a_2^{-\alpha} b_n^{-\alpha})$. One can extend with no difficulty (4.2) to the multivariate case, although the explicit expression becomes some what cumbersome.

Using Remark (3.5) we can generate the following two examples.

Example 4.3. Let $(T_1, \ldots, T_n)$ be given by (4.1) and let Z be a Weibull random variable with parameters μ and α independent of $(T_1, \ldots, T_n)$. The random vector $(Z^{-1}T_1, \ldots, Z^{-1}T_n)$ with a survival probability at $t_1, \ldots, t_n$, $(\min_i t_i > 0)$, given by

$$\mu\alpha[\mu + \sum_J \lambda_J \max_{i\varepsilon J} t_i^\alpha]^{-1} \tag{4.3}$$

satisfies (3.2).

Example 4.4. Let $(T_1, \ldots, T_n)$ be given by (4.1) and let Z be a positive random variable independent of $(T_1, \ldots, T_n)$ with a density equal to $e^{-\mu t^\alpha} t^{\beta-1} \alpha\mu^{\frac{\beta+1}{\alpha}-1} \Gamma^{-1}(\frac{\beta+1}{\alpha} - 1)$. The random vector $(Z^{-1}T_1, \ldots, Z^{-1}T_n)$ with a survival probability at $t_1, \ldots, t_n$, $(\min t_i > 0)$ equal to

$$\Gamma\mu(\mu + \sum_J \lambda_J \max_{i\varepsilon J} t_i^\alpha]^{\frac{\beta+1}{\alpha}-1} \tag{4.4}$$

satisfies (3.2).

Finally let $\mu_1, \ldots, \mu_n$ be positive real numbers, let β be in the set $(0, 1]$ and let α be in $[1, \infty)$. The survival probability given by

$$\exp[-\sum_{i=1}^n \mu_i t_i^\alpha]^\beta, \tag{4.5}$$

clearly has Weibull minima with $g(a_1, \ldots, a_n)$ equal to $(\sum_{i=1}^n \mu_i a_i^{-\alpha})^\beta$.

Essary and Marshall (1974) presented two bivariate distributions that have exponential marginals and the minima of the two components is exponential, however their joint distributions are not bivariate Marshall-Olkin (1967) exponential. The absolute continuous distribution given in (4.5) for $\beta = 1/\alpha$ is a simple n-dimensional example for such a situation.

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Let $T_1, \ldots, T_n$ be positive random variables with finite means. Further let I be the collection of all subsets of $\{1, \ldots, n\}$, and let ξ be a function from the nth Euclidian space to I, that equals to J, $(J \in I)$ at $(a_1, \ldots, a_n)$ iff $\min_{i\in J} a_iT_i < \min_{i\notin J} a_iT_i$. We prove that $\min_{1\le i\le n} a_iT_i$ and $\xi(a_1, \ldots, a_n)$ are independent random variables for every n real numbers $a_1, \ldots, a_n$ iff for every n positive real numbers $b_1, \ldots, b_n$ and $r = 1, \ldots, n$ the random variables $\min_{1\le i\le n} a_iT_i/E(\min_{1\le i\le n} a_iT_i)$ and T_r/ET_r are identically distributed. Further we provide an explicit formula for the distribution of $\xi(a_1, \ldots, a_n)$. Multivariate distributions that possess the independence property are presented. Their use in Reliability growth or decay models as well as in Mathematical Epidemiology are discussed.